Quadratic equations of projective $PGL_2(\mathbb{C})$-varieties.

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Abstract

We make explicit the equations of any projective $PGL_2(\mathbb{C})$-variety defined by quadrics. We study their zero-locus and their relationship with the geometry of the Veronese curve.

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Introduction.

Due to the progress of mathematical computer systems, like Maple, Macaulay2, Singular, Bertini and others, it is important to know explicitly the equations defining some known varieties. In this paper, we address this task for projective varieties stable under $PGL_2(\mathbb{C})$, the simplest of the simple Lie groups. In fact, we give all the quadratic equations of any projective variety stable under $PGL_2(\mathbb{C})$. We restrict ourselves to varieties inside $\mathbb{P}S^r(\mathbb{C}^2)$, where r is a natural number.

Let $r \geq 2$ be a natural number. Recall from [1] that the $\mathfrak{sl}_2(\mathbb{C})$-module $S^r(\mathbb{C}^2)$ is simple, that $S^r(\mathbb{C}^2) \cong S^r(\mathbb{C}^2)^\vee$ and that the decomposition of $S^2(S^r(\mathbb{C}^2))$ into simple submodules is given by

$$S^2(S^r(\mathbb{C}^2)) = \bigoplus_{m\geq 0} S^{2r-4m}(\mathbb{C}^2).$$

In this article, we investigate varieties $M_m \subseteq \mathbb{P}^r = \mathbb{P}S^r(\mathbb{C}^2)$ generated in degree two by $S^{2r-4m}(\mathbb{C}^2)^\vee$. Specifically, let $f_m : S^2(S^r(\mathbb{C}^2)) \to S^{2r-4m}(\mathbb{C}^2)$ be the projection and let

$$M_m = \{x \in \mathbb{P}S^r(\mathbb{C}^2) \mid f_m(xx) = 0\}.$$

If $f_m = (q_0, \ldots, q_{2r-4m})$, then the generators of the ideal of M_m are given by

$$\langle q_0, \ldots, q_{2r-4m} \rangle \cong S^{2r-4m}(\mathbb{C}^2)^\vee.$$

In the first section we study the equations defining M_m. In the second section we give a bound for the dimension of the variety M_m. It is unknown if it is irreducible. Any $PGL_2(\mathbb{C})$-variety X defined by quadrics is of the form

$$X = M_{m_1} \cap \ldots \cap M_{m_s}, \quad I(X)_2 = S^{2r-4m_1}(\mathbb{C}^2)^\vee \oplus \ldots \oplus S^{2r-4m_s}(\mathbb{C}^2)^\vee.$$

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Then the knowledge of the quadratic equations of M_m gives the explicit quadratic equations defining X. Also, the bound on the dimension of M_m gives a bound on the dimension of X.

Experiments and new theorems: at the beginning of the first section, we found recursive equations that define the quadrics containing the varieties M_m. We used a mathematical software (Maple) to compute the coefficients of theses quadrics. All the observations made were proven in Theorem 4, Proposition 5 and Lemma 7. We give a close formula for these coefficients.

Using the quadratic equations of the first section, we computed with another mathematical software (Macaulay2) the dimensions and the degrees of M_m. We proved in Proposition 8, Proposition 10, Example 11, Theorem 12 and Theorem 13 some patterns that emerge from computations.

1. Quadrics defining $M_m \subseteq \mathbb{P}^r$.

Let us fix a natural number r and a projection $f_m : S^2(S^r(\mathbb{C}^2)) \to S^{2r-4m}(\mathbb{C}^2)$. For simplicity, let us denote $f = f_m$. Let $n = 2r - 4m$ be a fixed even number.

Consider the following basis in $\mathfrak{sl}_2(\mathbb{C})$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $x_0 \in S^r(\mathbb{C}^2)$ and $w_0 \in S^n(\mathbb{C}^2)$ be maximal weight vectors. The action of $Y \in \mathfrak{sl}_2(\mathbb{C})$ on these vectors, generates bases $\{x_0, \ldots, x_r\}$ of $S^r(\mathbb{C}^2)$ and $\{w_0, \ldots, w_n\}$ of $S^n(\mathbb{C}^2)$. Specifically,

$$x_i = \frac{Y^i x_0}{i!}, \quad w_k = \frac{Y^k w_0}{k!}, \quad 0 \leq i \leq r, \quad 0 \leq k \leq n.$$

Using these bases, $f = \sum_0^n q_k w_k$, where $\{q_k\}_{k=0}^n$ are the quadratic equations of M_m.

Given that f is $\mathfrak{sl}_2(\mathbb{C})$-linear, we have the following relations:

$$Yf(x_ix_j) = f(Yx_ix_j) \Longleftrightarrow \sum_{k=0}^{n} q_k(x_ix_j)Yw_k = \sum_{k=0}^{n} q_k(Yx_ix_j)w_k \Longleftrightarrow$$

$$\sum_{k=0}^{n-1} q_k(x_ix_j)(k+1)w_{k+1} = \sum_{k=0}^{n} q_k((i+1)x_{i+1}x_j + (j+1)x_ix_{j+1})w_k \Longleftrightarrow$$

$$kq_{k-1}(x_ix_j) = (i+1)q_k(x_{i+1}x_j) + (j+1)q_k(x_ix_{j+1}), \quad 0 \leq k \leq n,\ 0 \leq i, j \leq r.$$

Note that all the forms depend recursively on q_n. In particular, if $q_n = 0$, the rest of the forms q_k are zero. Doing the same computation with X instead of Y, we get a similar recursion:

$$(n-k)q_{k+1}(x_ix_j) = (r-i+1)q_k(x_{i-1}x_j) + (r-j+1)q_k(x_ix_{j-1}), \quad 0 \leq k \leq n,\ 0 \leq i, j \leq r.$$

In these equations all the forms depend on q_0. With H we get conditions on each quadratic form,

$$Hf(x_ix_j) = f(Hx_ix_j) \Longleftrightarrow \sum_{k=0}^{n} q_k(x_ix_j)Hw_k = \sum_{k=0}^{n} q_k(Hx_ix_j) \Longleftrightarrow$$

$$\sum_{k=0}^{n} q_k(x_ix_j)(n-2k)w_k = \sum_{k=0}^{n} q_k((r-2i)x_ix_j + (r-2j)x_ix_j)w_k \Longleftrightarrow$$

$$(n-2k)q_k(x_ix_j) = (2r-2(i+j))q_k(x_ix_j) \Longleftrightarrow$$
$$(n-2k-2r+2i+2j)q_k(x_ix_j) = 0, \quad 0 \le k \le n,\ 0 \le i, j \le r.$$

Note that if $n-2r \neq 2k-2i-2j$, then $q_k(x_ix_j) = 0$. Saying thin in a different way, $q_k(x_ix_j) = 0$ except maybe for $j = 2m+k-i$.

Corollary 1. *Let r, n, $\{x_0,\dots,x_r\}$ and $\{w_0,\dots,w_n\}$ be as before and let q_0 be an arbitrary bilinear form on $S^r(\mathbb{C}^2)$ such that:*

$$0 = (i+1)q_0(x_{i+1},x_j) + (j+1)q_0(x_i,x_{j+1}), \quad (2r-2i-2j-n)q_0(x_i,x_j) = 0, \quad 0 \le i, j \le r.$$

Then there exists a unique $\mathfrak{sl}_2(\mathbb{C})$-morphism $f : S^r(\mathbb{C}^2)\otimes S^r(\mathbb{C}^2) \to S^n(\mathbb{C}^2)$ such that its component over w_0 is q_0. Even more, f is symmetric if and only if q_0 is symmetric.

Proof. Let i, j, k be three integers such that $0 \le k \le n$, $0 \le i, j \le r$. Assume we have defined q_k and let us define q_{k+1} using the recursive formula,

$$(n-k)q_{k+1}(x_i,x_j) = (r-i+1)q_k(x_{i-1},x_j) + (r-j+1)q_k(x_i,x_{j-1}).$$

Note that q_{k+1} is symmetric if and only if q_0 is symmetric. Let $f = q_0w_0 + \ldots + q_nw_n$. By construction it is a $\mathfrak{sl}_2(\mathbb{C})$-morphism and it is unique. $\square$

Corollary 2. *A quadratic form q_0 that extends to an $\mathfrak{sl}_2(\mathbb{C})$-map $f : S^2(S^r(\mathbb{C}^2)) \to S^{2r-4m}(\mathbb{C}^2)$, $f = q_0w_0 + \ldots + q_nw_n$, is given by*

$$q_0(x_ix_j) = \begin{cases} (-1)^i\binom{2m}{i}\lambda & \text{if } j = 2m-i \\ 0 & \text{otherwise} \end{cases}$$

where λ is a complex number. In particular, if $\lambda \in \mathbb{Q}$, all the coefficients of q_0 are rational. This implies that $q_k(x_ix_j) \in \mathbb{Q}$ for every $0 \le k \le n$ and $0 \le i, j \le r$.

Proof. Let us analyze in more detail the hypothesis on the quadratic form q_0 given in the previous corollary. The first condition,

$$0 = (i+1)q_0(x_{i+1}x_j) + (j+1)q_0(x_ix_{j+1}),$$

implies that q_0 depends only on the values $q_0(x_0x_j)$. This is because, given $q_0(x_0x_j)$ for every $0 \le j \le r$, we may define

$$q_0(x_1x_j) = -\frac{j+1}{2}q_0(x_0x_{j+1}).$$

Thus, if we have defined up to $q_0(x_ix_j)$ for some $0 < i < r$, we have

$$q_0(x_{i+1}x_j) = -\frac{j+1}{i+1}q_0(x_ix_{j+1}).$$

Let us discuss now the second hypothesis of the previous corollary,

$$(2r-2i-2j-n)q_0(x_ix_j) = 0.$$

Given that $n = 2r-4m$ we have $(2r-2i-2j-n) = 0$ if and only if $i+j = 2m$. Then

$$q_0(x_ix_j) \neq 0 \Longrightarrow i+j = 2m.$$

Let $\lambda = q_0(x_0 x_{2m})$ be arbitrary. Then applying the recursion we have

$$q_0(x_i x_{2m-i}) = (-1)^i \binom{2m}{i} \lambda, \quad 0 \le i \le 2m$$

$\square$

Corollary 3. *A $\mathfrak{sl}_2(\mathbb{C})$-linear map $f : S^2(S^r(\mathbb{C}^2)) \to S^{2r-4m}(\mathbb{C}^2)$ depends on one parameter, $\lambda \in \mathbb{C}$. In other words,*

$$\dim_{\mathfrak{sl}_2(\mathbb{C})}(S^2(S^r(\mathbb{C}^2)), S^{2r-4m}(\mathbb{C}^2)) = 1.$$

Proof. This fact is well known but in this case we are emphasizing the fact that every morphism depends just on one coefficient λ. $\square$

Now that we know exactly the coefficients of the quadratic form q_0, let us study the other forms, $\{q_1, \dots, q_n\}$.

First, we investigate the forms $\{q_1, \dots, q_{\frac{n}{2}}\}$. Then we prove that q_k and q_{n-k} are related $(0 \le k \le \frac{n}{2})$.

Theorem 4. *Let $\lambda = q_0(x_0 x_{2m})$ and $j = 2m + k - i$. Then for $0 \le k \le \frac{n}{2}$,*

$$\binom{n}{k} q_k(x_i x_j) = \lambda \sum_{s=\max(0,i-k)}^{\min(2m,i)} (-1)^s \binom{2m}{s} \binom{r-s}{r-i} \binom{r-2m+s}{r-j}$$

Proof. Recall these identities:

$$X x_i x_j = (r - i + 1) x_{i-1} x_j + (r - j + 1) x_j x_{j-1}.$$

$$X^s x_i = (r - i + 1)(r - i + 2) \dots (r - i + s) x_{i-s} = s! \binom{r-i+s}{r-i} x_{i-s}.$$

$$X^k x_i x_j = \sum_{l=0}^{k} \binom{k}{l} (X^l x_i)(X^{k-l} x_j).$$

From the equation $X f(x_i x_j) = f(X x_i x_j)$, we get

$$(n - k + 1) q_k(x_i x_j) = q_{k-1}(X x_i x_j).$$

Then

$$(n-k+1)(n-k+2)\dots(n) q_k(x_i x_j) = (n-k+2)\dots(n) q_{k-1}(X x_i x_j) =$$
$$= (n-k+3)\dots(n) q_{k-2}(X^2 x_i x_j) = \dots = q_0(X^k x_i x_j).$$

Without loss of generality we may assume $r > 2m$. When $r = 2m$ (i.e. $n = 0$) we obtain only q_0 that we already know (Corollary 2). Then

$$k! \binom{n}{k} q_k(x_i x_j) = q_0(X^k x_i x_j) = \sum_{l=0}^{k} \binom{k}{l} q_0(X^l x_i X^{k-l} x_j) =$$

$$= \sum_{l=0}^{k} \binom{k}{l} l! \binom{r-i+l}{r-i} (k-l)! \binom{r-j+k-l}{r-j} q_0(x_{i-l} x_{j-k+l}) =$$

$$= \sum_{l=0}^{k} \binom{k}{l} l! \binom{r-i+l}{r-i} (k-l)! \binom{r-j+k-l}{r-j} (-1)^{i-l} \binom{2m}{i-l} \lambda.$$

Dividing by $k!$, the binomial $\binom{k}{l}$ simplifies.

Finally, making the change of variable $s = i - l$, we get

$$\binom{n}{k} q_k(x_i x_j) = \lambda \sum_{s=i-k}^{i} (-1)^s \binom{2m}{s} \binom{r-s}{r-i} \binom{r-2m+s}{r-j}.$$

By convention, the binomials that do not make sense are zero. □

Let us prove now the relationship between the forms q_k and q_{n-k}.

Proposition 5. *Let k and i be two integers such that $0 \leq k \leq r - 2m$ and $0 \leq i \leq r$. Let $j = 2m + k - i$ and let $n = 2r - 4m$. Then*

$$q_k(x_i x_j) = q_{n-k}(x_{r-i} x_{r-j}).$$

Proof. Recall the three conditions obtained from the fact that f is $\mathfrak{sl}_2(\mathbb{C})$-linear,

$$k q_{k-1}(x_i x_j) = (i+1) q_k(x_{i+1} x_j) + (j+1) q_k(x_i x_{j+1}). \tag{1}$$

$$(n-k) q_{k+1}(x_i x_j) = (r-i+1) q_k(x_{i-1} x_j) + (r-j+1) q_k(x_i x_{j-1}). \tag{2}$$

$$(n-2k) q_k(x_i x_j) = (2r - 2(i+j)) q_k(x_i x_j). \tag{3}$$

Let us make the following change of variables in the second recursion, (2),

$$k' = n-k,\ i' = r-i,\ j' = r-j.$$

Note that $0 \leq k' \leq n/2$ and $0 \leq i', j' \leq r$. Then

$$k' q_{k'-1}(x_{i'} x_{j'}) = (i'+1) q_{k'}(x_{i'+1} x_{j'}) + (j'+1) q_{k'}(x_{i'} x_{j'+1}). \tag{2'}$$

Let $a_k(i,j) = q_k(x_i x_j)$ and $b_{k'}(i',j') = q_{k'}(x_{i'} x_{j'})$. Then

$$k a_{k-1}(i,j) = (i+1) a_k(i+1,j) + (j+1) a_k(i,j+1). \tag{1}$$

$$k b_{k-1}(i,j) = (i+1) b_k(i+1,j) + (j+1) b_k(i,j+1). \tag{2'}$$

Then the recursions are the same. If the initial data of these are equal, $a_{\frac{n}{2}} = b_{\frac{n}{2}}$, then $q_k(x_i x_j) = q_{n-k}(x_{r-i} x_{r-j})$.

$$a_{\frac{n}{2}}(i, 2m + \frac{n}{2} - i) = q_{\frac{n}{2}}(x_i x_{2m+\frac{n}{2}-i}) = q_{\frac{n}{2}}(x_i x_{2m+r-2m-i}) = q_{\frac{n}{2}}(x_i x_{r-i}) =$$

$$q_{\frac{n}{2}}(x_{r-i} x_i) = b_{\frac{n}{2}}(i, r-i) = b_{\frac{n}{2}}(i, 2m + \frac{n}{2} - i).$$

□

Corollary 6. *For every $0 \leq k \leq n/2$ we have $rk(q_k) = rk(q_{n-k}) \leq 2m + k + 1$.*

Proof. The matrix assigned to the quadratic form q_k has at least $2m+k+1$ nonzero coordinates. They appear in some anti-diagonal ($i+j=2m+k$) making nonzero rows linearly independent. □

In general, the equality does not hold. For example, if $r=6$ and $n=4$ (that is, $m=2$), then $q_2(x_1x_5)=q_2(x_5x_1)=0$ making the rank less than or equal to $2+4+1$. In this case, $\mathrm{rk}(q_0)=\mathrm{rk}(q_4)=5$, $\mathrm{rk}(q_1)=\mathrm{rk}(q_3)=6$ and $\mathrm{rk}(q_2)=5<7$.

Finally, let us give a lemma that we are going to use in the next section.

Lemma 7. *Let $\lambda=q_0(x_0x_{2m})\neq 0$ and let k be such that $0\leq k\leq n/2$. Then*

$$q_k(x_0x_{2m+k})=q_{n-k}(x_rx_{r-2m})\neq 0.$$

Even more, if $m=0$,

$$q_k(x_ix_{k-i})=q_{n-k}(x_{r-i}x_{r-k+i})\neq 0,\quad 0\leq i\leq r.$$

Proof. From Theorem 4 we have the formula

$$q_k(x_0x_{2m+k})=\lambda\frac{\binom{r-2m}{k}}{\binom{n}{k}}\neq 0.$$

And from Proposition 5, $q_{n-k}(x_rx_{r-2m})=q_k(x_0x_{2m+k})\neq 0$.

Similarly if $m=0$,

$$q_{n-k}(x_{r-i}x_{r-k+i})=q_k(x_ix_{k-i})=\lambda\frac{\binom{r}{r-i}\binom{r}{r-k+i}}{\binom{n}{k}}\neq 0,\quad 0\leq i\leq r.$$

□

2. Geometric properties of $M_m\subseteq\mathbb{P}^r$.

In the previous section we computed the equations for M_m. Recall that $M_m\subseteq\mathbb{P}S^r(\mathbb{C}^2)$ is a projective $PGL_2(\mathbb{C})$-variety generated in degree two by

$$\langle q_0,\dots,q_{2r-4m}\rangle\subseteq S^2(S^r(\mathbb{C}^2)^\vee).$$

In this section we use these equations to compute a bound for the dimension of M_m.

Let us introduce some new notation. Let

$$b_i^k(m)=b_i^k:=q_k(x_ix_{2m+k-i})=q_{n-k}(x_{r-i}x_{r-2m-k+i}),\quad 0\leq k\leq\frac{n}{2},\ 0\leq i\leq r.$$

Given that q_k is symmetric, we have $b_i^k=b_{2m+k-i}^k$.

If $x=a_0x_0+\ldots+a_rx_r$, then

$$q_k(a_0,\dots,a_r)=\sum_{i=0}^{2m+k}q_k(x_ix_{2m+k-i})a_ia_{2m+k-i}=\sum_{i=0}^{2m+k}b_i^ka_ia_{2m+k-i}.$$

$$q_{n-k}(a_0,\dots,a_r)=\sum_{i=0}^{2m+k}q_{n-k}(x_{r-i}x_{r-2m-k+i})a_{r-i}a_{r-2m-k+i}=\sum_{i=0}^{2m+k}b_i^ka_{r-i}a_{r-2m-k+i}.$$

With this notation, let us write the derivatives of q_k with respect to a_i,

$$\frac{\partial q_k(a_0,\dots,a_r)}{\partial a_i} = b_i^k a_{2m+k-i} + b_{2m+k-i}^k a_{2m+k-i} = 2b_i^k a_{2m+k-i}.$$

Proposition 8. *The variety $M_m \subseteq \mathbb{P}^r$ has dimension* $\dim(M_m) < 2m$. *If $m = 0$, $M_m = \emptyset$.*

Proof. Let us compute the rank of the Jacobian matrix of

$$(a_0,\dots,a_r) \to (q_0(a_0,\dots,a_r),\dots,q_n(a_0,\dots,a_r)).$$

It is a $(n+1)\times(r+1)$-matrix.

$$\begin{pmatrix}
b_0^0 a_{2m} & b_1^0 a_{2m-1} & \dots & b_{2m}^0 a_0 & 0 & 0 & 0 & \dots & 0 \\
b_0^1 a_{2m+1} & \dots & \dots & \dots & b_{2m+1}^1 a_0 & 0 & 0 & \dots & 0 \\
b_0^2 a_{2m+2} & \dots & \dots & \dots & \dots & b_{2m+2}^2 a_0 & 0 & \dots & 0 \\
 & & & \vdots & & & & & \\
b_0^{r-2m} a_r & b_1^{r-2m} a_{r-1} & \dots & \dots & \dots & \dots & \dots & \dots & b_r^{r-2m} a_0 \\
0 & b_0^{r-2m-1} a_r & b_1^{r-2m-1} a_{r-1} & \dots & \dots & \dots & \dots & \dots & b_{r-1}^{r-2m-1} a_1 \\
0 & 0 & b_0^{r-2m-2} a_r & b_1^{r-2m-2} a_{r-1} & \dots & \dots & \dots & \dots & b_{r-2}^{r-2m-2} a_2 \\
 & & & \vdots & & & & & \\
0 & 0 & \dots & \dots & 0 & b_0^0 a_r & b_1^0 a_{r-1} & \dots & b_{2m}^0 a_{r-2m}
\end{pmatrix}$$

Let Z be the hyperplane given by $\{a_r = 0\}$. From Lemma 7, we know that $b_0^k \neq 0$ for $0 \le k \le r-2m$. Then for every point not in Z, the last $r-2m+1$ rows of the previous matrix are linearly independent making the rank greater that or equal to $r-2m+1$. If $m = 0$, the rank is $r+1$.

Take X an irreducible component of M_m. It is also a $PGL_2(\mathbb{C})$-variety. Recall that the closure of an orbit must contain orbits of lesser dimension. In particular, X must contain a closed orbit. The unique closed orbit of $PGL_2(\mathbb{C})$ in $\mathbb{P}S^r(\mathbb{C}^2)$ is the orbit of the maximal weight vector, x_0, [1, Claim 23.52]. Using the equivariant isomorphism $S^r(\mathbb{C}^2) \cong S^r(\mathbb{C}^2)^\vee$, the vector x_r corresponds to the maximal weight vector of $\mathbb{P}S^r(\mathbb{C}^2)^\vee$. Then its orbit is closed in $\mathbb{P}S^r(\mathbb{C}^2)^\vee$. Applying the isomorphism again, we obtain a closed orbit in $\mathbb{P}S^r(\mathbb{C}^2)$, hence the orbit of x_r is equal to the orbit of x_0. This implies that the point corresponding to x_r, $(0 : \dots : 0 : 1)$ is in X, hence $X \setminus Z$ is non-empty. Then a generic smooth point of X has dimension less than $2m$. □

Notation 9. Our intention now is to relate the geometry of the Veronese curve with the geometry of M_m. This analysis gives a lower bound for the dimension of M_m.

Recall briefly the definition of the Veronese curve $c_r \subseteq \mathbb{P}^r$ and its osculating varieties $T^p c_r$. The Veronese curve may be given parametrically (over an open affine subset) by

$$c_r : t \longrightarrow (1, t, t^2, \dots, t^r).$$

Its tangential variety, denoted $T^1 c_r$, may be given by

$$(t, \lambda_1) \longrightarrow c_r + \lambda_1 c_r'.$$

It depends on two parameters. One indicates the point in the curve and the other, the tangent vector on that point.

In general, its p-osculating variety, $T^p c_r$ is given by

$$(t, \lambda_1, \dots, \lambda_p) \longrightarrow c_r + \lambda_1 c_r' + \dots + \lambda_p c_r^{(p)}.$$

In each point of the curve, stands a p-dimensional plane.

We consider the curve c_r and its osculating varieties $T^p c_r$ inside $\mathbb{P}^r$. The dimensions of c_r and of $T^p c_r$ are the expected, $p+1$.

In the article [3], the author computed the Hilbert polynomials of the varieties $T^p c_r$,

$$H_{T^p c_r}(d) = (dr - dp + 1)\binom{p+d}{d} - (dr - dp + d - 1)\binom{p+d-1}{d}.$$

This implies that $\dim(T^p c_r) = p+1$, $\deg(c_r) = r$ and $\deg(T^1 c_r) = 2(r-1)$.

Proposition 10. *The variety M_m contains $T^{m-1}c_r$ but does not contain $T^m c_r$. In particular, $\dim(M_m) \geq m$.*

Proof. This proposition follows from [1, Exercise 11.32]. It says that

$$I(T^p c_r)_2 \cong \bigoplus_{\alpha \geq p+1} S^{2r-4\alpha}(\mathbb{C}^2).$$

Given that $S^{2r-4m}(\mathbb{C}^2) \subseteq I(T^{m-1}c_r)_2$ we get $I(M_m) \subseteq I(T^{m-1}c_r)$.

Similarly, if $I(M_m)_2 \subseteq I(T^m c_r)_2$, then $S^{2r-4m}(\mathbb{C}^2) \subseteq I(T^m c_r)_2$. A contradiction. □

Example 11. Suppose that r is even and that $m = r/2$. Then we have exactly one equation q_0. It is a quadratic form whose matrix (diagonal of rank $r+1$) has coefficients $\lambda(-1)^i\binom{r}{i}$. In fact this is the only quadric in $\mathbb{P}^r$ invariant under $PGL_2(\mathbb{C})$. For $r = 4$ this quadric is well known, [2, 10.12].

The variety $M_m = \mathbb{P}\{q_0 = 0\} \subseteq \mathbb{P}^r$ is a quadric of maximal rank, hence irreducible. Being a hypersurface, it has $\dim(M_m) = r-1$. Then, by Proposition 10, we obtain

$$\begin{cases} T^{\frac{r}{2}-1}c_r \subsetneq M_m & \text{if } r > 2. \\ c_2 = M_m & \text{if } r = 2. \end{cases}$$

With this example we deduce that the dimension of M_m may be strictly greater than m.

Theorem 12. *If $r \geq 3$ is odd and $m = (r-1)/2$, then M_m has codimension 3 and degree 8.*

Proof. We know that $I(M_m) = \langle q_0, q_1, q_2 \rangle$ where

$$q_0(a_0, \dots, a_r) = b_0^0 a_0 a_{r-1} + b_1^0 a_1 a_{r-2} + \ldots + b_{r-1}^0 a_{r-1} a_0,$$
$$q_1(a_0, \dots, a_r) = b_0^1 a_0 a_r + b_1^1 a_1 a_{r-1} + \ldots + b_r^1 a_r a_0,$$
$$q_2(a_0, \dots, a_r) = b_0^0 a_r a_1 + b_1^0 a_{r-1} a_2 + \ldots + b_{r-1}^0 a_1 a_r.$$

The coefficients of the quadratic forms satisfy the following relations

$$b_0^0 = b_{r-1}^0, \quad b_1^0 = b_{r-2}^0, \quad \ldots, \quad b_{m-1}^0 = b_{m+1}^0,$$
$$b_0^1 = b_r^1, \quad b_1^1 = b_{r-1}^1, \quad \ldots, \quad b_{m-1}^1 = b_{m+2}^1, \quad b_m^1 = b_{m+1}^1.$$

To see that the dimension is $r-3$ let us compute the rank of the Jacobian matrix at a specific point $p \in M_m$. The Jacobian matrix is given by

$$\begin{pmatrix} b_0^0 a_{r-1} & b_1^0 a_{r-2} & \ldots & b_{r-1}^0 a_0 & 0 \\ b_0^1 a_r & b_1^1 a_{r-1} & \ldots & b_{r-1}^1 a_1 & b_r^1 a_0 \\ 0 & b_0^0 a_r & \ldots & b_{r-2}^0 a_2 & b_{r-1}^0 a_1 \end{pmatrix}.$$

Let $p = (p_0 : \ldots : p_r) \in \mathbb{P}^r$ be a point such that

$$p_i = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $q_0(p) = q_1(p) = q_2(p) = 0$, hence $p \in M_m$. The Jacobian matrix at p is equal to

$$\begin{pmatrix} 0\ldots0 & b^0_{r-m} & 0 & 0 & 0\ldots0 & b^0_{r-1} & 0 \\ 0\ldots0 & 0 & b^1_{r-m+1} & 0 & 0\ldots0 & 0 & b^1_r \\ 0\ldots0 & 0 & 0 & b^0_{r-m+1} & 0\ldots0 & 0 & 0 \end{pmatrix}.$$

Given that $b^0_i \neq 0$ for all $0 \leq i \leq r$ (see Corollary 2) and that $b^1_r = b^1_0 \neq 0$ (see Lemma 7) the previous matrix has maximal rank, hence the codimension of M_m at p is equal to 3. This implies that the codimension of M_m is 3 and the degree is 8.

Note that the point p is in $T^{m-1}c_r$ and that the points on the curve c_r are singular. □

Theorem 13. *If $r \geq 8$ is even and $m = r/2 - 1$, then M_m has codimension 5 and degree 32.*

Proof. Let us argue as in the proof of Theorem 12. We know that $I(M_m) = \langle q_0, \ldots, q_4 \rangle$,

$$q_0(a_0, \ldots, a_r) = \sum_{i=0}^{r-2} b^0_i a_i a_{r-2-i}, \quad q_1(a_0, \ldots, a_r) = \sum_{i=0}^{r-1} b^1_i a_i a_{r-1-i},$$

$$q_2(a_0, \ldots, a_r) = \sum_{i=0}^{r} b^2_i a_i a_{r-i},$$

$$q_3(a_0, \ldots, a_r) = \sum_{i=0}^{r-1} b^1_i a_{r-i} a_{i+1}, \quad q_4(a_0, \ldots, a_r) = \sum_{i=0}^{r-2} b^0_i a_{r-i} a_{i+2}.$$

Let $p = (p_0 : \ldots : p_r) \in \mathbb{P}^r$ be a point such that

$$p_i = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $p \in M_m$. The Jacobian matrix at p is equal to

$$\begin{pmatrix} 0\ldots0 & b^0_{r-m-1} & 0 & 0 & 0 & 0 & 0\ldots0 & b^0_{r-2} & 0 & 0 \\ 0\ldots0 & 0 & b^1_{r-m} & 0 & 0 & 0 & 0\ldots0 & 0 & b^1_{r-1} & 0 \\ 0\ldots0 & 0 & 0 & b^2_{r-m+1} & 0 & 0 & 0\ldots0 & 0 & 0 & b^2_r \\ 0\ldots0 & 0 & 0 & 0 & b^1_{r-m+1} & 0 & 0\ldots0 & 0 & 0 & 0 \\ 0\ldots0 & 0 & 0 & 0 & 0 & b^0_{r-m+1} & 0\ldots0 & 0 & 0 & 0 \end{pmatrix}.$$

From Corollary 2 and Lemma 7, we know that b^2_0, b^1_0, b^0_{r-2} and b^0_{r-m+1} are non-zero numbers. But given that $b^2_r = b^2_0$ and $b^1_{r-1} = b^1_0$, they are also non-zero. We need to prove that b^1_{r-m+1} is non-zero for $r \geq 8$. Recall that $b^1_{r-m+1} = b^1_{m-2}$.

$$b^1_{m-2} \neq 0 \Longleftrightarrow \binom{n}{1} q_1(x_{m-2}x_{m+3}) \neq 0 \Longleftrightarrow \sum_{s=m-3}^{m-2} (-1)^s \binom{2m}{s} \binom{r-s}{r-m+2} \binom{r-2m+s}{r-m-3} \neq 0 \Longleftrightarrow$$

$$\binom{2m}{m-3}(r-m+3) - \binom{2m}{m-2}(r-m-2) \neq 0 \iff \frac{m-2}{m+3} \neq \frac{r-m-2}{r-m+3} \iff$$
$$(m-2)(r-m+3) - (r-m-2)(m+3) \neq 0 \iff 10m - 5r \neq 0 \iff 2m \neq r.$$

Given that $2m = r - 2$, we obtain $b^1_{r-m+1} \neq 0$. □

Example 14. We computed the dimension and the degree of M_m for several values of r and m:

$m\backslash r$	2	3	4	5	6	7	8	9	10	11	12	13
1	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{1}$
2			$\underline{3}$	$\underline{2}$	3	2	2	2	2	2	3	2
3					$\underline{5}$	$\underline{4}$	$\underline{3}$	3	5	3	3	3
4							$\underline{7}$	$\underline{6}$	$\underline{5}$	4	4	4
5									$\underline{9}$	$\underline{8}$	$\underline{7}$	6
6											$\underline{11}$	$\underline{10}$

Table: Dimension of $M_m \subseteq \mathbb{P}^r$.

$m\backslash r$	2	3	4	5	6	7	8	9	10	11	12	13
1	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$	$\underline{9}$	$\underline{10}$	$\underline{11}$	$\underline{12}$	$\underline{13}$
2			$\underline{2}$	$\underline{8}$	5	12	14	16	18	20	22	24
3					$\underline{2}$	$\underline{8}$	$\underline{32}$	21	12	27	30	33
4							$\underline{2}$	$\underline{8}$	$\underline{32}$	128	36	40
5									$\underline{2}$	$\underline{8}$	$\underline{32}$	128
6											$\underline{2}$	$\underline{8}$

Table: Degree of $M_m \subseteq \mathbb{P}^r$.

The numbers underlined are known in general (see Example 11, Theorem 12, Theorem 13). Recall also that $m \leq \dim M_m < 2m$.

Remark 15. To end this section, let us make a little remark and some more computations. Suppose now that we want to study the variety X defined by the quadrics that contain $T^p c_r$. In other words, X is generated in degree two and $I(X)_2 = I(T^p c_r)_2$.

Given that c_r is generated in degree two, when $p = 0$, we have the equality, $X = c_r$. In the general case, $T^p c_r \subseteq X$.

From Proposition 8 and the fact that $X = M_{p+1} \cap \ldots \cap M_{\lfloor r/2 \rfloor}$, we get

$$p + 1 \leq \dim(X) \leq 2p + 1.$$

We computed the dimension of the variety X in the case $I(X)_2 = I(T^p c_r)_2$:

	$\mathbb{P}^4$	$\mathbb{P}^5$	$\mathbb{P}^6$	$\mathbb{P}^7$	$\mathbb{P}^8$	$\mathbb{P}^9$	$\mathbb{P}^{10}$	$\mathbb{P}^{11}$	$\mathbb{P}^{12}$	$\mathbb{P}^{13}$
$I(T^1 c_r)_2$	$\underline{3}$	$\underline{2}$	2	2	2	2	2	2	2	2
$I(T^2 c_r)_2$			$\underline{5}$	$\underline{4}$	3	3	3	3	3	3
$I(T^3 c_r)_2$					$\underline{7}$	$\underline{6}$	4	4	4	4
$I(T^4 c_r)_2$							$\underline{9}$	$\underline{8}$	6	5
$I(T^5 c_r)_2$									$\underline{11}$	$\underline{10}$

The dimensions underlined are those in which $I(T^p c_r)_2 = I(M_m)_2$ for some m, so it is information from a previous table.

In the variety 4-osculating of $c_{12} \subseteq \mathbb{P}^{12}$ the pattern breaks. The dimension is 6 instead of 5. We deduce that this variety is not generated in degree two.

Assume now that $5 \leq r \leq 8$. Let X_r be the variety generated in degree two by $I(T^1 c_r)_2$. We computed that X_r is irreducible, $\dim(X_r) = 2$ and $\deg(X_r) = 2(r-1)$. Then we know explicitly the equations defining $T^1 c_5$, $T^1 c_6$, $T^1 c_7$ and $T^1 c_8$ (set-theoretically).

$$I(X_5) = \langle x_5x_0 - 3x_4x_1 + 2x_3x_2, x_4x_0 - 4x_3x_1 + 3{x_2}^2, x_5x_1 - 4x_4x_2 + 3{x_3}^2\rangle.$$

$$I(X_6) = \langle x_4x_0 - 4x_3x_1 + 3{x_2}^2, x_6x_0 - 9x_4x_2 + 8{x_3}^2, x_6x_2 - 4x_5x_3 + 3{x_4}^2,$$
$$x_5x_0 - 3x_4x_1 + 2x_3x_2, x_6x_1 - 3x_5x_2 + 2x_4x_3, x_6x_0 - 6x_5x_1 + 15x_4x_2 - 10{x_3}^2\rangle.$$

$$I(X_7) = \langle x_7x_3 - 4x_6x_4 + 3x_5^2, 2x_7x_3 + x_6x_4 - 3x_5^2, x_7x_2 + 3x_6x_3 - 4x_5x_4, x_3x_0 - x_2x_1,$$
$$x_4x_0 - 4x_3x_1 + 3x_2^2, x_5x_0 + 3x_4x_1 - 4x_3x_2, x_7x_4 - x_6x_5, 2x_4x_0 + x_3x_1 - 3x_2^2,$$
$$x_5x_0 - 3x_4x_1 + 2x_3x_2, x_6x_0 - 6x_5x_1 + 15x_4x_2 - 10x_3^2, x_6x_0 - x_5x_1 - 5x_4x_2 + 5x_3^2,$$
$$x_6x_0 + 8x_5x_1 + x_4x_2 - 10x_3^2, x_7x_0 + 5x_6x_1 - 21x_5x_2 + 15x_4x_3, x_7x_0 + 23x_6x_1 + 51x_5x_2 - 75x_4x_3,$$
$$x_7x_1 + 8x_6x_2 + x_5x_3 - 10x_4^2, x_7x_1 - x_6x_2 - 5x_5x_3 + 5x_4^2, x_7x_1 - 6x_6x_2 + 15x_5x_3 - 10x_4^2,$$
$$x_7x_2 - 3x_6x_3 + 2x_5x_4, x_7x_0 - 5x_6x_1 + 9x_5x_2 - 5x_4x_3, x_2x_0 - x_1^2, x_7x_5 - x_6^2\rangle.$$

$$I(X_8) = \langle x_4x_0 - 4x_3x_1 + 3x_2^2, x_8x_2 - 6x_7x_3 + 15x_6x_4 - 10x_5^2, x_8x_4 - 4x_7x_5 + 3x_6^2,$$
$$x_8x_1 + 2x_7x_2 - 12x_6x_3 + 9x_5x_4, x_8x_3 - 3x_7x_4 + 2x_6x_5, 3x_6x_0 - 4x_5x_1 - 11x_4x_2 + 12x_3^2,$$
$$x_5x_0 - 3x_4x_1 + 2x_3x_2, x_7x_0 + 2x_6x_1 - 12x_5x_2 + 9x_4x_3, x_7x_0 - 5x_6x_1 + 9x_5x_2 - 5x_4x_3,$$
$$x_8x_1 - 5x_7x_2 + 9x_6x_3 - 5x_5x_4, x_6x_0 - 6x_5x_1 + 15x_4x_2 - 10x_3^2,$$
$$x_8x_0 + 12x_7x_1 - 22x_6x_2 - 36x_5x_3 + 45x_4^2, 3x_8x_2 - 4x_7x_3 - 11x_6x_4 + 12x_5^2,$$
$$x_8x_0 - 2x_7x_1 - 8x_6x_2 + 34x_5x_3 - 25x_4^2, x_8x_0 - 8x_7x_1 + 28x_6x_2 - 56x_5x_3 + 35x_4^2\rangle.$$

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